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A note on Bernstein and Markov type inequalities

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Abstract

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. We prove that for any polynomial p of degree at most n

$$\max_{z\in\partial\mathbb{D}}\left|\frac{p(z)-p(\bar{z})}{z-\bar{z}}\right| \leqslant n \max_{0\leqslant j\leqslant n}\left|\frac{p\left(e^{ij\pi/n}\right)+p\left(e^{-ij\pi/n}\right)}{2}\right|,$$

where $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. We show how this result is related to classical inequalities of Bernstein and Markov and to more recent results due to Duffin and Schaeffer.

1. Introduction

Let $\mathcal{P}_n$ be the class of polynomials $p(z) := \sum_{k=0}^{n} a_k z^k$ of degree at most n with complex coefficients. We write, together with $\mathbb{D} := \{z : |z| < 1\}$,

$$\|p\|_{\mathbb{D}} := \max_{z\in\partial\mathbb{D}} |p(z)|$$

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and

$$\|p\|_{[-1,1]} := \max_{-1\leqslant x\leqslant 1} |p(x)|.$$

According to the famous inequalities of Bernstein and Markov we have

$$\|p'\|_{\mathbb{D}} \leqslant n\|p\|_{\mathbb{D}}, \tag{1}$$

$$\|p'\|_{[-1,1]} \leqslant n^2\|p\|_{[-1,1]}. \tag{2}$$

We refer the reader to the survey paper of Bojanov [1], and to the recent book by Rahman and Schmeisser [7] for up-to-date references concerning (1) and (2) and their numerous extensions.

In this paper, we shall be concerned by the following *discrete* refinements of (1) and (2):

$$\|p'\|_{[-1,1]} \leqslant n^2 \max_{0\leqslant j\leqslant n} |p(\cos(j\pi/n))| \tag{3}$$

and

$$\|p'\|_{\mathbb{D}} \leqslant n \max_{0\leqslant j\leqslant 2n-1} \left|p\left(e^{ij\pi/n}\right)\right|. \tag{4}$$

Inequality (3) is a result due to Duffin and Schaeffer [4]; it is known that equality holds there only for multiples of the nth Chebyshev polynomial $T_n \in \mathcal{P}_n$ which is defined for $x \in [-1, 1]$ by $T_n(x) := \cos(n \arccos(x))$. Inequality (4) is due to Frappier et al. [5]; it is further possible [3] to prove that equality holds in (4) only for the monomials $p_n(z) := Kz^n$, where $K \in \mathbb{C}$.

Each polynomial $p \in \mathcal{P}_n$ possesses a unique expansion in terms of Chebyshev polynomials as

$$p(z) = \sum_{k=0}^{n} A_k(p)T_k(z) \quad (A_k(p) \in \mathbb{C},\ 0\leqslant k\leqslant n).$$

We clearly have

$$p(\cos\theta) = \sum_{k=0}^{n} A_k(p)\cos k\theta \quad (\theta \in \mathbb{R})$$

and

$$p'(\cos\theta) = \sum_{k=0}^{n} A_k(p)\, k\frac{\sin k\theta}{\sin\theta} \quad (\theta \in \mathbb{R}).$$

We can therefore, by considering the polynomial $p_*(z) := \sum_{k=0}^{n} A_k(p)\, z^k \in \mathcal{P}_n$, formulate inequality (3) of Duffin and Schaeffer as

$$\left|\frac{e^{i\theta}p_*'\left(e^{i\theta}\right) - e^{-i\theta}p_*'\left(e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}}\right| \leqslant n^2 \max_{0\leqslant j\leqslant n}\left|\frac{p_*\left(e^{ij\pi/n}\right) + p_*\left(e^{-ij\pi/n}\right)}{2}\right|,$$

for all $\theta \in \mathbb{R}$.

The main results of this paper are the following:

Theorem 1. *Let $p \in \mathcal{P}_n$. Then*

$$\max_{\theta \in \mathbb{R}} \left| \frac{p\left(e^{i\theta}\right) - p\left(e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}} \right| \leqslant n \max_{0 \leqslant j \leqslant n} \left| \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2} \right|.$$

Theorem 2. *Let $p \in \mathcal{P}_n$ and $\theta \in \mathbb{R}$. Then*

$$\left| p'\left(e^{i\theta}\right) \right| \leqslant n \max_{j \in J_n} \left| \frac{p\left(e^{i(\theta + j\pi/n)}\right) + p\left(e^{i(\theta - j\pi/n)}\right)}{2} \right|,$$

where $J_n := \{0\} \cup \{j : 1 \leqslant j \leqslant n,\ j \text{ odd}\}$.

Theorem 1 is in the spirit of (4). It gives an upper bound for the uniform norm over $\mathbb{R}$ of the divided difference

$$\frac{p\left(e^{i\theta}\right) - p\left(e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}}$$

of a polynomial $p \in \mathcal{P}_n$. Our proof shows that

$$\left| \frac{p\left(e^{i\theta}\right) - p\left(e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}} \right| < n \max_{0 \leqslant j \leqslant n} \left| \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2} \right|,$$

when θ is not an integer multiple of π and $p \not\equiv 0$. Theorem 2 is clearly an improvement of Bernstein's inequality (1); moreover the cardinality of J_n is much smaller than that of the set $\{0, 1, \ldots, 2n - 1\}$ and therefore Theorem 2 may yield an estimate better than (4). It is possible to identify all polynomials $p \in \mathcal{P}_n$ which satisfy

$$|p'(1)| = n \max_{j \in J_n} \left| \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2} \right|. \tag{5}$$

Details are supplied in [3]. Let us simply point out here that the set of all polynomials satisfying (5) with $n = 3$ is

$$\{p : p(z) = (2M + b)z^3 + (M - b)z^2 + (M - b)z - (M - b)\},$$

where M and b are arbitrary complex numbers.

2. Proofs of Theorems 1 and 2

Let $\mathcal{T}_n$ denote the vector space of all trigonometric polynomials of degree at most n. The following quadrature formula (we mention [2, Theorem 2.1] as a ready reference) turns out to be useful.

Lemma 1. *The quadrature formula*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} t(\theta)\,d\theta = \frac{1}{m}\sum_{j=0}^{m-1} t\left(\frac{2j\pi}{m}+\gamma\right) \quad (\gamma \in \mathbb{R})$$

holds for all $t \in \mathcal{T}_{m-1}$ and γ real.

We shall prove that, given $\theta \in [0,\ \pi]$, there exist real numbers

$$\alpha_j = \alpha_j(\theta), \quad j = 0, \ldots, n,$$

such that

$$\frac{p\left(e^{i\theta}\right) - p\left(e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}} = \sum_{j=0}^{n} (-1)^j \alpha_j \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2} \tag{6}$$

for all $p \in \mathcal{P}_n$, and

$$\sum_{j=0}^{n} |\alpha_j(\theta)| \leqslant n. \tag{7}$$

The above representation is linear and it is clearly sufficient to consider the basic polynomials $p(z) = z^k,\ 0 \leqslant k \leqslant n$. Thus, we are naturally led to the system of equations

$$\sum_{j=0}^{n} (-1)^j \cos\left(\frac{kj\pi}{n}\right)\alpha_j = \frac{\sin k\theta}{\sin \theta}, \quad k = 0, 1, \ldots, n. \tag{8}$$

Let us consider the vector space $\mathcal{T}_{n,e}$ of all *even* trigonometric polynomials of the form $t(\varphi) = \sum_{k=0}^{n} a_k(t) \cos k\varphi$. The polynomials τ_j defined by

$$\tau_j(\varphi) = \frac{\prod_{v=0, v\neq j}^{n} (\cos \varphi - \cos v\pi/n)}{\prod_{v=0, v\neq j}^{n} (\cos j\pi/n - \cos v\pi/n)}, \quad j = 0, \ldots, n \tag{9}$$

belong to $\mathcal{T}_{n,e}$ (see [6, pp. 19–22]) and satisfy the relations

$$\tau_j\,(v\pi/n) = \delta_{j,v}, \quad j, v = 0, \ldots, n,$$

where $\delta_{j,v}$ represents the usual Kronecker's delta. The function set

$$\{\cos k\varphi\ (k = 0, \ldots, n), \quad \varphi \in [0, \pi]\}$$

is a Chebyshev system. Therefore, we obtain the interpolation formula

$$t(\varphi) = \sum_{j=0}^{n} t\,(j\pi/n)\ \tau_j(\varphi) \quad (t \in \mathcal{T}_{n,e}).$$

Then, the quadrature formula

$$\int_0^\pi t(\varphi)\,\mu(\varphi)\,d\varphi = \sum_{j=0}^{n} \alpha_j\, t\left(\frac{j\pi}{n}\right),$$

with coefficients

$$\alpha_j := \int_0^\pi \tau_j(\varphi)\,\mu(\varphi)\,d\varphi \quad (0 \leqslant j \leqslant n)$$

holds for all $t \in \mathcal{T}_{n,e}$ and any function μ, integrable over $[0, \pi]$. In particular,

$$\sum_{j=0}^{n} \cos\left(\frac{kj\pi}{n}\right) \alpha_j = \int_0^\pi \cos(k\varphi)\,\mu(\varphi)\,d\varphi \quad (k = 0, \ldots, n)$$

and because of (7) and (8) we would like to choose $\mu_\theta := \mu$ such that

$$\int_0^\pi \cos(k\varphi)\,\mu_\theta(\varphi)\,d\varphi = \frac{\sin(n-k)\theta}{\sin\,\theta} \quad (k = 0, \ldots, n) \tag{10}$$

and

$$\sum_{j=0}^{n} |\alpha_j| = \sum_{j=0}^{n} \left| \int_0^\pi \tau_j(\varphi)\,\mu_\theta(\varphi)\,d\varphi \right| \leqslant n. \tag{11}$$

The system $\{\cos\,k\varphi\ (k = 0, 1, \ldots)\}$ is a complete orthogonal system in $L^2[0, \pi]$. So, if $\mu_\theta \in L^2[0, \pi]$, then

$$\mu_\theta(\varphi) = \sum_{v=0}^{\infty} c_v(\theta) \cos\,v\varphi \quad \text{(a.e.)}.$$

We shall look for μ_θ of the form

$$\mu_\theta(\varphi) := \sum_{v=0}^{n} c_v(\theta) \cos\,v\varphi \quad \text{(a.e.)}.$$

Relations (10) uniquely determine μ_θ as

$$\mu_\theta(\varphi) = \frac{1}{\pi}\left\{ \frac{\sin\,n\theta}{\sin\,\theta} + 2\sum_{v=1}^{n-1} \frac{\sin(n-v)\theta}{\sin\,\theta} \cos\,v\varphi \right\}.$$

Clearly, we may think of μ_θ as *a θ-perturbation of the classical Fejér's kernel*.

The basic interpolation polynomials (9) are uniquely determined. Simple computations show that

$$\tau_0(\varphi) = \frac{\sin\,n\varphi \cos\,\varphi/2}{2n \sin\,\varphi/2}, \quad \tau_n(\varphi) = \frac{(-1)^{n+1} \sin\,n\varphi \sin\,\varphi/2}{2n \cos\,\varphi/2} \tag{12}$$

and

$$\tau_j(\varphi) = \frac{(-1)^j \sin n\varphi \sin \varphi}{2n \sin\left(\frac{\varphi}{2} + \frac{j\pi}{2n}\right) \sin\left(\frac{\varphi}{2} - \frac{j\pi}{2n}\right)}, \quad 1 \leqslant j \leqslant n-1. \tag{13}$$

Our next task is to compute the coefficients $\alpha_j(\theta)$. We claim

$$\alpha_0 = \frac{1}{2n} \frac{1-\cos n\theta}{1-\cos \theta}, \quad \alpha_n = \frac{(-1)^{n-1}}{2n} \frac{1-(-1)^n \cos n\theta}{1+\cos \theta} \tag{14}$$

and for $1 \leqslant j \leqslant n-1$,

$$\alpha_j = \frac{(-1)^j - \cos n\theta}{n\left(\cos \frac{j\pi}{n} - \cos \theta\right)}. \tag{15}$$

Indeed, when $1 \leqslant j \leqslant n-1$

$$\alpha_j = \int_0^\pi \tau_j(\varphi)\, \mu_\theta(\varphi)\, d\varphi = \frac{1}{2} \int_{-\pi}^{\pi} \frac{(-1)^j}{n} \frac{\sin \varphi \sin n\varphi}{\cos \frac{j\pi}{n} - \cos \varphi} \mu_\theta(\varphi)\, d\varphi.$$

and observing that in the above integral the integrand is a trigonometric polynomial of degree $2n-1$, we apply Lemma 1 with $m = 2n$ and $\gamma = 0$ in order to obtain

$$\alpha_j = \frac{(-1)^j}{2n \sin \theta} \left[4 \sum_{s=1}^{n-1} \cos\left(\frac{sj\pi}{n}\right) \sin s\theta + 2\,(-1)^j \sin n\theta \right]$$

and (15) follows upon applications of elementary identities. Similarly, (12) may be used to obtain (14). We finally prove (11): let $P_n(z) := \frac{1-z^n}{1-z}$. By (14) and (15) with $z = e^{i\gamma_0}$

$$\begin{aligned}
\sum_{j=0}^{n} \left|\alpha_j(\gamma_0)\right| &= \frac{|P_n(z)|^2}{2n} + \sum_{j=1}^{n-1} \frac{\left|P_n\left(ze^{ij\pi/n}\right)\right| \left|P_n\left(ze^{-ij\pi/n}\right)\right|}{n} + \frac{|P_n(-z)|^2}{2n} \\
&\leqslant \frac{|P_n(z)|^2}{2n} + \sum_{j=1}^{n-1} \frac{\left|P_n\left(ze^{ij\pi/n}\right)\right|^2 + \left|P_n\left(ze^{-ij\pi/n}\right)\right|^2}{2n} + \frac{|P_n(-z)|^2}{2n} \\
&= \frac{1}{2n} \sum_{j=0}^{2n-1} \left|P_n(\mu_j z)\right|^2,
\end{aligned}$$

where $\{\mu_j\}_{j=0}^{2n-1}$ is the set of distinct $2n$-roots of unity. Another application of Lemma 1 with $m = 2n$ and $\gamma = \gamma_0$ yields

$$\frac{1}{2n} \sum_{j=0}^{2n-1} |P_n(\mu_j z)|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1-\cos(n\,\theta)}{1-\cos(\theta)}\, d\theta = n$$

and (11) follows. This completes the proof of (6).

Theorem 1 is an immediate consequence of (6) and (7). Theorem 2, with $\theta = 0$, follows by letting $\theta \to 0$ in (6) and (7) upon noticing that

$$\alpha_j(0) \neq 0$$

only when $j \in J_n$. The general case of Theorem 2 follows by considering $p_\theta(z) := p(e^{i\theta}z)$ for given $p \in \mathcal{P}_n$.

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